

TRIMMING A GORENSTEIN IDEAL

LARS WINTHER CHRISTENSEN, OANA VELICHE, AND JERZY WEYMAN

ABSTRACT. Let Q be a regular local ring of dimension 3. We show how to trim a Gorenstein ideal in Q to obtain an ideal that defines a quotient ring that is close to Gorenstein in the sense that its Koszul homology algebra is a Poincaré duality algebra P padded with a non-zero graded vector space on which $P_{\geq 1}$ acts trivially. We explicitly construct an infinite family of such rings.

1. INTRODUCTION

Let Q be a regular local ring with maximal ideal \mathfrak{n} . Quotient rings of Q that have projective dimension at most 3 as Q -modules have been classified based on the multiplicative structure of their Koszul homology algebras. To be precise, let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal such that the minimal free resolution of $R = Q/\mathfrak{a}$ over Q has length at most 3. By a result of Buchsbaum and Eisenbud [4], the resolution carries a structure of an associative differential graded commutative algebra, and based on that structure Avramov, Kustin, and Miller [3] and Weyman [9] established a classification in terms of the induced multiplicative structure on $\mathrm{Tor}_*^Q(R, \mathbb{k})$, where \mathbb{k} is the residue field of Q . Finally, as graded \mathbb{k} -algebras, the Koszul homology algebra of R and $\mathrm{Tor}_*^Q(R, \mathbb{k})$ are isomorphic; see Avramov [1] for an in-depth treatment.

An ideal $\mathfrak{a} \subset Q$ is called *Gorenstein* if the quotient $R = Q/\mathfrak{a}$ is a Gorenstein ring. By a classic result of Avramov and Golod [2], a Gorenstein ring is characterized by the fact that its Koszul homology algebra $A = H(K^R)$ has Poincaré duality. In the classification mentioned above, a Gorenstein ring that is not complete intersection belongs to a parametrized family $\mathbf{G}(r)$, where r is the rank of the canonical map

$$\delta: A_2 \longrightarrow \mathrm{Hom}_{\mathbb{k}}(A_1, A_3);$$

see [1, 1.4.2]. It was conjectured in [1] that all rings of class $\mathbf{G}(r)$ are Gorenstein, but Christensen and Veliche [5] gave sporadic examples of rings of class $\mathbf{G}(r)$ that are not Gorenstein. In this paper we present a systematic construction and achieve:

(1.1) Theorem. *Let Q be the power series algebra in three variables over a field. For every $r \geq 3$ there is quotient ring of Q that is of class $\mathbf{G}(r)$ and not Gorenstein.*

The quotient rings in Theorem (1.1) are obtained as follows: Let \mathfrak{n} be the maximal ideal of Q and start with a graded Gorenstein ideal $\mathfrak{g} \subseteq \mathfrak{n}^2$ generated by $2m+1$

Date: 14 October 2016.

2010 Mathematics Subject Classification. 13C99; 13H10.

Key words and phrases. Gorenstein ring, Koszul homology, Poincaré duality algebra.

This work is part of a body of research that started during the authors' visit to MSRI in Spring 2013 and continued during a months-long visit by L.W.C. to Northeastern University; the hospitality of both institutions is acknowledged with gratitude. L.W.C. was partly supported by NSA grant H98230-11-0214, and J.W. was partly supported by NSF DMS grant 1400740.

elements. Trim \mathfrak{g} by replacing one minimal generator g by $\mathfrak{n}g$; this removes a 1-dimensional subspace from \mathfrak{g} . The quotient of Q by the resulting ideal is a ring of type 2; in particular, it is not Gorenstein, and for $m \geq 3$ it is of class $\mathbf{G}(r)$. Theorem (1.1) is consequence of Proposition (3.5), which builds on a more general but slightly less precise statement about local rings, Theorem (2.4).

2. LOCAL RINGS

Let Q be a d -dimensional regular local ring with maximal ideal \mathfrak{n} and residue field \mathbb{k} . For an ideal \mathfrak{a} in Q , we denote by $\mu(\mathfrak{a})$ the minimal number of generators of \mathfrak{a} . Let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal and set $R = Q/\mathfrak{a}$. We denote by K^R the Koszul complex on a minimal set of generators for the maximal ideal $\mathfrak{n}/\mathfrak{a}$ of R ; one has $K^R = R \otimes_Q K^Q$. The Koszul complex is an exterior algebra, and the homology algebra $A = H(K^R)$ is a graded-commutative \mathbb{k} -algebra. Denote by c the projective dimension of R as a Q -module; by the Auslander–Buchsbaum Formula and depth sensitivity of the Koszul complex one has $c = \max\{i \mid A_i \neq 0\}$. The number $\text{rank}_{\mathbb{k}}(A_c)$ is called the *type* of R . If the ideal \mathfrak{a} is \mathfrak{n} -primary, then one has $c = d$ and the type of R is the socle rank, i.e. $\text{type}(R) = \text{rank}_{\mathbb{k}}(0 :_R \mathfrak{n}/\mathfrak{a})$.

(2.1) **Classification.** Let Q be as above, and let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal such that $R = Q/\mathfrak{a}$ has projective dimension 3 as a Q -module. The possible multiplicative structures on the graded-commutative \mathbb{k} -algebra $A = H(K^R) \cong \text{Tor}_*^Q(R, \mathbb{k})$ were identified in [3]. By assumption one has $A_{\geq 4} = 0$, and the possible structures are described by the invariants

$$p = \text{rank}_{\mathbb{k}}(A_1 \cdot A_1), \quad q = \text{rank}_{\mathbb{k}}(A_1 \cdot A_2), \quad \text{and} \quad r = \text{rank}_{\mathbb{k}}(A_2 \xrightarrow{\delta} \text{Hom}_{\mathbb{k}}(A_1, A_3)).$$

From [1, thm. 3.1] one extracts the following description of all the possible classes of rings that are not Gorenstein.

(2.1.1)	Class	p	q	r	Restrictions
	\mathbf{B}	1	1	2	
	$\mathbf{G}(r)$	0	1	r	$2 \leq r \leq \mu(\mathfrak{a}) - 2$
	$\mathbf{H}(p, q)$	p	q	q	$q \leq \text{type}(R)$
	\mathbf{T}	3	0	0	

In [3] the multiplication tables for the different structures are given. In particular, if $R = Q/\mathfrak{a}$ is a ring of class $\mathbf{G}(r)$, then with $m = \mu(\mathfrak{a})$ and $t = \text{type}(R)$ there exist bases for A_1 , A_2 , and A_3 :

$$\mathbf{e}_1, \dots, \mathbf{e}_m, \quad \mathbf{f}_1, \dots, \mathbf{f}_{m+t-1}, \quad \text{and} \quad \mathbf{g}_1, \dots, \mathbf{g}_t$$

such that the only non-zero products are $\mathbf{e}_i \mathbf{f}_i = \mathbf{g}_1 = -\mathbf{f}_i \mathbf{e}_i$ for $1 \leq i \leq r$. That is, the subalgebra P of A spanned by $1, \mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{f}_1, \dots, \mathbf{f}_r$, and \mathbf{g}_1 is a pure Poincaré duality algebra, in the sense that the only non-trivial products are those from the perfect pairing. Moreover, $P_{\geq 1}$ acts trivially on the rest of A .

The next result is proved in [6]; the argument is based on linkage theory and cannot be reproduced here without significant overhead.

(2.2) **Proposition.** *Let (Q, \mathfrak{n}) be a regular local ring and let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be a perfect ideal of grade 3 that is minimally generated by 5 elements and not Gorenstein. If, with the notation above, the ring Q/\mathfrak{a} has $p = 0$, then it has $r \leq 1$. \square*

(2.3) **Lemma.** *Let (Q, \mathfrak{n}) be a regular local ring and consider an \mathfrak{n} -primary ideal $\mathfrak{g} \subseteq \mathfrak{n}^2$, minimally generated by elements g_0, \dots, g_k . Let s_1, \dots, s_t be elements of Q whose classes in Q/\mathfrak{g} form a basis for the socle. The ideal $\mathfrak{a} = \mathfrak{n}g_0 + (g_1, \dots, g_k)$ is \mathfrak{n} -primary and the classes of g_0, s_1, \dots, s_t in Q/\mathfrak{a} form a basis for the socle; in particular one has $\text{type}(Q/\mathfrak{a}) = \text{type}(Q/\mathfrak{g}) + 1$.*

Proof. As \mathfrak{g} is \mathfrak{n} -primary, it follows from the containment $\mathfrak{n}\mathfrak{g} \subseteq \mathfrak{a}$ that \mathfrak{a} is \mathfrak{n} -primary. Consider the rings $R = Q/\mathfrak{a}$ and $S = Q/\mathfrak{g}$; there is an exact sequence

$$0 \longrightarrow \mathfrak{g}/\mathfrak{a} \longrightarrow R \longrightarrow S \longrightarrow 0,$$

and an isomorphism of Q -modules $\mathfrak{g}/\mathfrak{a} \cong \mathbb{k}$, where \mathbb{k} is the residue field of Q . Tensoring with the Koszul complex K^Q one gets an exact sequence of Q -complexes,

$$(*) \quad 0 \longrightarrow \mathbb{k} \otimes_Q K^Q \xrightarrow{\alpha} K^R \xrightarrow{\beta} K^S \longrightarrow 0.$$

Let d be the dimension of Q . From the sequence in homology associated to $(*)$ one gets the following exact sequence

$$0 \longrightarrow \mathbb{k} \xrightarrow{H_d(\alpha)} H_d(K^R) \xrightarrow{H_d(\beta)} H_d(K^S).$$

The rings R and S are artinian, and a rank count yields

$$\text{type}(R) = \text{rank}_{\mathbb{k}}(H_d(K^R)) \leq \text{rank}_{\mathbb{k}}(H_d(K^S)) + 1 = \text{type}(S) + 1.$$

It is clear that the residue classes $[g_0]$ and $[s_1], \dots, [s_t]$ in R are non-zero socle elements. Moreover, they are \mathbb{k} -linearly independent: Indeed, the elements $[s_1], \dots, [s_t]$ are \mathbb{k} -linearly independent, because of the inclusion $\mathfrak{a} \subset \mathfrak{g}$. Further, suppose one has $[g_0] = \sum_{i=1}^t [u_i][s_i]$ where the elements u_i are units in Q . It follows that $g_0 - \sum_{i=1}^t u_i s_i$ is in $\mathfrak{a} \subseteq \mathfrak{g}$, and as $g_0 \in \mathfrak{g}$ one gets $\sum_{i=1}^t u_i s_i \in \mathfrak{g}$, a contradiction. Thus, there are $t + 1$ \mathbb{k} -linearly independent elements in the socle of R . \square

For the next result, recall from work of J. Watanabe [8] that a grade 3 Gorenstein ideal in a regular ring is minimally generated by an odd number of elements.

(2.4) **Theorem.** *Let (Q, \mathfrak{n}) be a regular local ring of dimension 3 and let $\mathfrak{g} \subseteq \mathfrak{n}^2$ be an \mathfrak{n} -primary Gorenstein ideal minimally generated by elements g_0, \dots, g_{2m} . The ideal $\mathfrak{a} = \mathfrak{n}g_0 + (g_1, \dots, g_{2m})$ is \mathfrak{n} -primary, one has $\text{type}(Q/\mathfrak{a}) = 2$ and:*

- (a) *If $m = 1$, then $\mu(\mathfrak{a}) = 5$ and Q/\mathfrak{a} is of class **B**.*
- (b) *If $m = 2$, then one of the following holds:*
 - $\mu(\mathfrak{a}) = 4$ and Q/\mathfrak{a} is of class **H**(3, 2).
 - $\mu(\mathfrak{a}) = 5$ and Q/\mathfrak{a} is of class **B**.
 - $\mu(\mathfrak{a}) \in \{6, 7\}$ and Q/\mathfrak{a} is of class **G**(r) with $\mu(\mathfrak{a}) - 2 \geq r \geq \mu(\mathfrak{a}) - 3$.
- (c) *If $m \geq 3$, then Q/\mathfrak{a} is of class **G**(r) with $\mu(\mathfrak{a}) - 2 \geq r \geq \mu(\mathfrak{a}) - 3$.*

Proof. It follows from Lemma (2.3) that \mathfrak{a} is \mathfrak{n} -primary and $R = Q/\mathfrak{a}$ has type 2; in particular, R is not Gorenstein. Note that one has

$$2m \leq \mu(\mathfrak{a}) \leq 2m + 3.$$

Set $S = Q/\mathfrak{g}$; there is an exact sequence of Q -modules

$$0 \longrightarrow \mathfrak{g}/\mathfrak{a} \longrightarrow R \longrightarrow S \longrightarrow 0$$

and an isomorphism $\mathfrak{g}/\mathfrak{a} \cong \mathbb{k}$. Tensor with the Koszul complex K^Q to get an exact sequence of Q -complexes,

$$(*) \quad 0 \longrightarrow \mathbb{k} \otimes_Q K^Q \xrightarrow{\alpha} K^R \xrightarrow{\beta} K^S \longrightarrow 0,$$

where β is a morphism of DG Q -algebras. Set $A = H(K^R)$ and $G = H(K^S)$, one has

$$\begin{aligned} \text{rank}_{\mathbb{k}}(G_0) = 1 = \text{rank}_{\mathbb{k}}(G_3) \quad \text{and} \quad \text{rank}_{\mathbb{k}}(G_1) = 2m + 1 = \text{rank}_{\mathbb{k}}(G_2) \\ \text{rank}_{\mathbb{k}}(A_0) = 1, \text{rank}_{\mathbb{k}}(A_1) = \mu(\mathfrak{a}), \text{rank}_{\mathbb{k}}(A_2) = \mu(\mathfrak{a}) + 1, \text{ and } \text{rank}_{\mathbb{k}}(A_3) = 2, \\ \text{and consider the exact sequence in homology associated to } (*) \end{aligned}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{k} & \xrightarrow[\quad 1 \quad]{H_3(\alpha)} & A_3 & \xrightarrow[\quad 1 \quad]{H_3(\beta)} & G_3 \longrightarrow \\ & & & & & & \downarrow 0 \\ & & \mathbb{k}^3 & \xrightarrow[\quad 3 \quad]{H_2(\alpha)} & A_2 & \xrightarrow[\mu(\mathfrak{a})-2]{H_2(\beta)} & G_2 \longrightarrow \\ & & & & & & \downarrow 2m+3-\mu(\mathfrak{a}) \\ & & \mathbb{k}^3 & \xrightarrow[\mu(\mathfrak{a})-2m]{H_1(\alpha)} & A_1 & \xrightarrow[2m]{H_1(\beta)} & G_1 \longrightarrow \\ & & & & & & \downarrow 1 \\ & & \mathbb{k} & \xrightarrow[\quad 0 \quad]{H_0(\alpha)} & A_0 & \xrightarrow[\quad 1 \quad]{H_0(\beta)} & G_0 \longrightarrow 0 \end{array}$$

where the numbers below the arrows indicate the ranks of the maps. As $H(\beta)$ is a homomorphism of graded \mathbb{k} -algebras, there is a commutative diagram

$$\begin{array}{ccc} G_2 & \xrightarrow[\quad 2m+1 \quad]{\delta_G} & \text{Hom}_{\mathbb{k}}(G_1, G_3) \\ \uparrow H_2(\beta) \quad \mu(\mathfrak{a})-2 & & \downarrow 2m \quad \text{Hom}_{\mathbb{k}}(H_1(\beta), G_3) \\ A_2 & \xrightarrow{\quad \varepsilon \quad} & \text{Hom}_{\mathbb{k}}(A_1, G_3) \\ \parallel & & \uparrow \text{Hom}_{\mathbb{k}}(A_1, H_3(\beta)) \\ A_2 & \xrightarrow[\quad r \quad]{\delta_A} & \text{Hom}_{\mathbb{k}}(A_1, A_3) \end{array}$$

The rank of δ_A is at least the rank of $\varepsilon = \text{Hom}_{\mathbb{k}}(A_1, H_3(\beta)) \circ \delta_A$. The rank of ε is at least $(\mu(\mathfrak{a}) - 2) - 1$ as δ_G is an isomorphism, see [1, 1.4.2], and the kernel of $\text{Hom}_{\mathbb{k}}(H_1(\beta), G_3)$ has rank 1. Thus, one has $r = \text{rank}(\delta_A) \geq \mu(\mathfrak{a}) - 3$.

In case $\mu(\mathfrak{a}) \geq 6$, one has $r \geq 3$, and since the type of R is 2, this implies that R is of class $\mathbf{G}(r)$; see (2.1.1). This proves part (c) and the last case of part (b). For $m = 2$ one has $4 \leq \mu(\mathfrak{a}) \leq 7$. If $\mu(\mathfrak{a}) = 5$ it follows from Proposition (2.2) and (2.1.1) that R is of class \mathbf{B} . If $\mu(\mathfrak{a}) = 4$, then R is of class $\mathbf{H}(3, 2)$ by [1, 3.4.2.(a)]. Finally, part (a) is a result of Faucett [7]. \square

3. A FAMILY OF GRADED LOCAL RINGS OF CLASS $\mathbf{G}(r)$

A grade 3 Gorenstein ideal of a local ring is by a result of Buchsbaum and Eisenbud [4, thm. 2.1] minimally generated by the sub-maximal Pfaffians of a $(2m + 1) \times (2m + 1)$ skew-symmetric matrix. Thus, skew-symmetric matrices are a source of

Gorenstein rings and, via Theorem (2.4), also a source of rings of class $\mathbf{G}(r)$ that are not Gorenstein. In this section, we construct an infinite family of such rings.

(3.1) Let \mathbb{k} be a field and set $Q = \mathbb{k}[[x, y, z]]$; let m be a positive integer.

Denote by U_m the $m \times m$ matrix over Q whose i^{th} row has entries

$$u_{i,m-i} = x, \quad u_{i,m-i+1} = z, \quad \text{and} \quad u_{i,m-i+2} = y$$

and 0 elsewhere; set

$$d_{-1} = 0, \quad d_0 = 1, \quad \text{and} \quad d_m = \det(U_m).$$

That is,

$$U_1 = [z], \quad U_2 = \begin{bmatrix} x & z \\ z & y \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & x & z \\ x & z & y \\ z & y & 0 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 0 & 0 & x & z \\ 0 & x & z & y \\ x & z & y & 0 \\ z & y & 0 & 0 \end{bmatrix}, \quad \dots$$

$$d_1 = z, \quad d_2 = xy - z^2, \quad d_3 = 2xyz - z^3, \quad d_4 = -3xyz^2 + x^2y^2 + z^4, \quad \dots$$

Notice that for every i in the range $2, \dots, m$ one has,

$$(3.1.1) \quad U_m = \left[\begin{array}{c|c} O_x & U_{i-1} \\ \hline U_{m-i+1} & {}^yO \end{array} \right],$$

where O_x is the appropriately sized matrix with x in the lower right corner and 0 elsewhere, and yO is the matrix with y in the top left corner and 0 elsewhere.

Let V_m be the $(2m+1) \times (2m+1)$ skew-symmetric matrix given by

$$(3.1.2) \quad V_m = \left[\begin{array}{c|c|c} O & O_x & U_m \\ \hline -(O_x)^T & 0 & {}^yO \\ \hline -U_m & -({}^yO)^T & O \end{array} \right],$$

where O is the $m \times m$ zero-matrix and, as above, O_x and yO are appropriately sized matrices with 0 everywhere but in the lower left and upper right corner, respectively. That is,

$$(3.1.3) \quad V_1 = \begin{bmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 & 0 & x & z \\ 0 & 0 & x & z & y \\ 0 & -x & 0 & y & 0 \\ -x & -z & -y & 0 & 0 \\ -z & -y & 0 & 0 & 0 \end{bmatrix}, \quad \dots$$

The sub-maximal Pfaffians of V_m are determined (up to a sign) by minors, $\text{pf}_i(V_m)^2 = \det((V_m)_{ii})$. Consider the ideal of Q generated by these Pfaffians,

$$(3.1.4) \quad \mathfrak{g}_m = (\text{pf}_1(V_m), \dots, \text{pf}_{2m+1}(V_m)).$$

(3.2) **Lemma.** *In the notation from (3.1) the next equalities hold for every $m \geq 1$.*

$$d_m = (-1)^{m-1} z d_{m-1} + x y d_{m-2} \quad \text{and}$$

$$d_m = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} (-1)^{\lfloor \frac{m-2j}{2} \rfloor} x^j y^j z^{m-2j}.$$

Proof. Per (3.1.1) with $i = 2$, expansion of the determinant of U_m along the first row yields

$$d_m = (-1)^m x \det((U_m)_{1,m-1}) + (-1)^{m+1} z \det(U_{m-1}).$$

From (3.1.1) with $i = 3$ it follows that expansion along the last column yields

$$\det((U_m)_{1,m-1}) = (-1)^m y \det(U_{m-2}).$$

Combining these two expressions, one gets the first equality. The second equality now follows by induction. \square

Evidently, the ideal \mathfrak{g}_m from (3.1.4) is contained in \mathfrak{n}^m ; in fact, one has $\mathfrak{g}_1 = \mathfrak{n}$. One can check that, though the generating matrices are different, the family of ideals $\{\mathfrak{g}_m\}_{m \geq 2}$ is the same as that provided by [4, prop. 6.2]. To understand what happens when one trims these ideals, we provide a more detailed description.

(3.3) Proposition. *Adopt the notation from (3.1) and let \mathfrak{n} denote the maximal ideal of Q . For every $m \geq 2$ the ideal $\mathfrak{g}_m \subseteq \mathfrak{n}^2$ is an \mathfrak{n} -primary Gorenstein ideal minimally generated by the elements*

$$x^{m-i}d_i \text{ and } y^{m-i}d_i \text{ for } 0 \leq i \leq m-1 \text{ and } d_m.$$

The ring Q/\mathfrak{g}_m has socle generated by the class of $x^{m-1}y^{m-1}$ and Hilbert series

$$\text{Hilb}_{Q/\mathfrak{g}_m}(t) = \sum_{i=0}^{m-2} \binom{i+2}{2} (t^i + t^{2m-2-i}) + \binom{m+1}{2} t^{m-1}.$$

Proof. Per (3.1.3) the Pfaffians of V_1 are, up to signs,

$$\text{pf}_1(V_1) = y = yd_0, \quad \text{pf}_2(V_1) = z = d_1, \quad \text{and} \quad \text{pf}_3(V_1) = x = xd_0.$$

For $m \geq 2$ we argue that, up to signs, one has

$$\begin{aligned} \text{pf}_i(V_m) &= y^{m-i+1}d_{i-1} \text{ for } 1 \leq i \leq m, \\ \text{pf}_{m+1}(V_m) &= d_m, \text{ and} \\ \text{pf}_{2m+2-i}(V_m) &= x^{m-i+1}d_{i-1} \text{ for } 1 \leq i \leq m. \end{aligned}$$

First notice that the equality $\text{pf}_{m+1}(V_m) = d_m$ is immediate from (3.1.2). Further, note that by symmetry in x and y it is sufficient to prove that $\text{pf}_i(V_m) = y^{m-i+1}d_{i-1}$ holds for $1 \leq i \leq m$. To compute $\text{pf}_1(V_m)$ notice that the matrix $(V_m)_{11}$ is a $2m \times 2m$ -matrix with $\pm y$ on the anti-diagonal and zeros below it. Thus, one has $\text{pf}_1(V_m) = y^m = y^m d_0$. Now, for i in the range $2, \dots, m$ consider the matrix $(V_m)_{ii}$ as 2×2 block matrix with blocks of size $m \times m$,

$$(V_m)_{ii} = \left[\begin{array}{c|c} X & W_i \\ \hline -W_i^T & O \end{array} \right],$$

where O is as in (3.1.2), i.e. it is zero. Thus, one has

$$\det((V_m)_{ii}) = \left| \begin{array}{c|c} X & W_i \\ \hline -W_i^T & O \end{array} \right| = (-1)^m \left| \begin{array}{c|c} W_i & X \\ \hline O & -W_i^T \end{array} \right| = (\det(W_i))^2.$$

Next, notice that W_i is obtained from U_m by removing row i and adding a row yO at the bottom. Thus, per (3.1.1) it has the form

$$W_i = \left[\begin{array}{c|c} O_x & U_{i-1} \\ \hline Y & O \end{array} \right],$$

where Y is the matrix obtained from U_{m-i+1} by removing the first row and adding a row yO at the bottom. In particular, it is a $(m-i+1) \times (m-i+1)$ -matrix with $\pm y$ on the anti-diagonal and zeros below it. Thus, computing the determinant of W_i by successive expansion on the last $m-i+1$ rows one gets, up to a sign, $\text{pf}_i(V_m) = y^{m-i+1}d_{i-1}$. It follows that \mathfrak{g}_m is generated by the listed elements.

The elements x^m, y^m, d_m form a Q -regular sequence in \mathfrak{g}_m , so it follows from [4, thm. 2.1] that \mathfrak{g}_m is a Gorenstein ideal minimally generated by the listed elements. In particular, \mathfrak{g}_m is \mathfrak{n} -primary. In fact, in this case it is elementary to see that the generating set is minimal: Notice from Lemma (3.2) that d_i is a linear combination of monomials of the form $x^j y^j z^{i-2j}$. Hence, each generator $x^{m-i}d_i$ is a linear combination of monomials of the form $x^{m-i+j}y^j z^{i-2j}$ while the generators $y^{m-i}d_i$ are linear combinations of monomials $x^j y^{m-i+j} z^{i-2j}$. Thus the generators are linear combinations of disjoint sets of degree m monomials and hence linearly independent.

The Hilbert series of the power series ring Q is $\text{Hilb}_Q(t) = \sum_{i=0}^{\infty} \binom{i+2}{2} t^i$. Since \mathfrak{g}_m is Gorenstein and minimally generated by $2m+1$ elements of degree m , the Hilbert series of the ring $S_m = Q/\mathfrak{g}_m$ is symmetric and given by

$$\text{Hilb}_{S_m}(t) = \sum_{i=0}^{m-2} \binom{i+2}{2} (t^i + t^{2m-2-i}) + \binom{m+1}{2} t^{m-1}.$$

In particular, the socle degree of S_m is $2m-2$. Evidently, one has $(x^{m-1}y^{m-1})\mathfrak{n} \subseteq \mathfrak{g}_m$, so it is sufficient to show that the element $x^{m-1}y^{m-1}$ is not in \mathfrak{g}_m , i.e. that it yields a non-zero socle element in S_m . If it were in \mathfrak{g}_m , then one would have $x(x^{m-2}y^{m-1})$ in \mathfrak{g}_m along with $x^{m-2}(y^m d_0) = y(x^{m-2}y^{m-1})$ and $x^{m-2}(y^{m-1}d_1) = z(x^{m-2}y^{m-1})$. Thus, $x^{m-2}y^{m-1}$ would yield a socle element in S_m of degree $2m-3$, whence it must be 0; i.e. one would have $x^{m-2}y^{m-1} \in \mathfrak{g}_m$. Reiterating this argument, one arrives at the conclusion that y^{m-1} is in \mathfrak{g}_m , which is absurd as the generators of \mathfrak{g}_m have degree m . \square

Finally, we apply the trimming procedure from Theorem (2.4) to the ideals \mathfrak{g}_m .

(3.4) Adopt the notation from (3.1). By Proposition (3.3) one has

$$\mathfrak{g}_2 = (x^2, xz, xy - z^2, yz, y^2).$$

Trimming the generators xz and yz one gets the following ideals of Q ,

$$\begin{aligned} (x, y, z)xz + (x^2, xy - z^2, yz, y^2) &= (x^2, xy - z^2, yz, y^2) \quad \text{and} \\ (x, y, z)yz + (x^2, xz, xy - z^2, y^2) &= (x^2, xz, xy - z^2, y^2). \end{aligned}$$

They are both minimally generated by 4 elements, so they define quotient rings of class $\mathbf{H}(3, 2)$; see Theorem (2.4)(b). Moreover, one has

$$\begin{aligned} (x, y, z)x^2 + (xz, xy - z^2, yz, y^2) &= (x^3, xz, xy - z^2, yz, y^2), \\ (x, y, z)y^2 + (x^2, xz, xy - z^2, yz) &= (x^2, xz, xy - z^2, yz, y^3), \quad \text{and} \\ (x, y, z)(xy - z^2) + (x^2, xz, yz, y^2) &= (x^2, xz, z^3, yz, y^2), \end{aligned}$$

so by Theorem (2.4)(b) these ideals define rings of class \mathbf{B} .

From the next result one immediately gets the statement of Theorem (1.1) about existence of infinite families of rings of class $\mathbf{G}(r)$ that are not Gorenstein.

(3.5) **Proposition.** *Adopt the notation from (3.1) and let \mathfrak{n} denote the maximal ideal of Q . Let g be one of the generators of \mathfrak{g}_m listed in (3.3), let \mathfrak{b} be the ideal generated by the remaining $2m$ generators of \mathfrak{g}_m , and set $\mathfrak{a} = \mathfrak{n}g + \mathfrak{b}$. For $m \geq 3$ the ring $R = Q/\mathfrak{a}$ has the following properties.*

- (a) *R is an artinian local ring of type 2 with socle generated by the classes of the elements g and $x^{m-1}y^{m-1}$.*
- (b) *If g is $x^{m-i}d_i$ or $y^{m-i}d_i$ for some $i \in \{1, \dots, m-1\}$, then \mathfrak{a} is minimally generated by $2m$ elements and R is of class $\mathbf{G}(2m-3)$.*
- (c) *If g is x^m , y^m , or d_m , then \mathfrak{a} is minimally generated by $2m+1$ elements and R is of class $\mathbf{G}(2m-2)$.*

Proof. Fix $m \geq 3$; for brevity the class in R or $S = Q/\mathfrak{g}_m$ of an element u in Q is also written u .

Part (a) is immediate from Lemma (2.3). We prove parts (b) and (c) together. First we describe the generators of \mathfrak{a} using the recurrence formula from Lemma (3.2). For $1 \leq i \leq m$ one has

$$(1) \quad \begin{aligned} x(x^{m-i}d_i) &= x^{m-(i-1)}((-1)^{i-1}zd_{i-1} + xyd_{i-2}) \\ &= (-1)^{i-1}z(x^{m-(i-1)}d_{i-1}) + y(x^{m-(i-2)}d_{i-2}). \end{aligned}$$

For $0 \leq i \leq m-2$ one has

$$(2) \quad \begin{aligned} y(x^{m-i}d_i) &= x^{m-(i+1)}(xyd_i) \\ &= x^{m-(i+1)}(d_{i+2} - (-1)^{i+1}zd_{i+1}) \\ &= x(x^{m-(i+2)}d_{i+2}) + (-1)^i z(x^{m-(i+1)}d_{i+1}) \quad \text{and moreover} \\ y(xd_{m-1}) &= x(yd_{m-1}). \end{aligned}$$

For $0 \leq i \leq m-1$ one has

$$(3) \quad \begin{aligned} z(x^{m-i}d_i) &= x^{m-i}(-1)^i(d_{i+1} - xyd_{i-1}) \\ &= (-1)^i x(x^{m-(i+1)}d_{i+1}) - (-1)^i y(x^{m-(i-1)}d_{i-1}). \end{aligned}$$

For $g = x^{m-i}d_i$ with $1 \leq i \leq m-1$ it follows immediately from (1)–(3) that $\mathfrak{n}g$ is contained in \mathfrak{b} , so $\mathfrak{a} = \mathfrak{b}$ is minimally generated by $2m$ elements. By symmetry the same is true for $g = y^{m-i}d_i$ with $1 \leq i \leq m-1$.

For $g = x^m$ one has $yg \in \mathfrak{b}$ and $zg \in \mathfrak{b}$ by (2) and (3), so \mathfrak{a} is generated by the $2m$ generators of \mathfrak{b} and x^{m+1} . To see that this is a minimal set of generators, note that the generators of \mathfrak{b} have degree m and none of them includes the term x^m . The statement for $g = y^m$ follows by symmetry.

For $g = d_m$ one has $xg \in \mathfrak{b}$ by (1) and $yg \in \mathfrak{b}$ by symmetry. Thus \mathfrak{a} is generated by the $2m$ generators of \mathfrak{b} and zd_m . To see that this is a minimal set of generators, note from Lemma (3.2) that zd_m has a z^{m+1} term, while the generators of \mathfrak{b} have degree m and none of them has a z^m term.

To determine the multiplicative structure on $A = H(K^R)$ we first describe a basis for A_1 . The Koszul complex K^R is the exterior algebra of the free R -module with basis $\{\varepsilon_x, \varepsilon_y, \varepsilon_z\}$ endowed with the differential given by $\partial(\varepsilon_x) = x$, $\partial(\varepsilon_y) = y$,

and $\partial(\varepsilon_z) = z$. We suppress the wedge in products on K^R and adopt the following shorthands

$$\varepsilon_{xy} = \varepsilon_x \varepsilon_y, \quad \varepsilon_{xz} = \varepsilon_x \varepsilon_z, \quad \varepsilon_{yz} = \varepsilon_y \varepsilon_z, \quad \text{and} \quad \varepsilon_{xyz} = \varepsilon_x \varepsilon_y \varepsilon_z.$$

Because of the symmetry in x and y we only consider $g = x^{m-i}d_i$. Given the minimal generating set of \mathfrak{a} described above, one gets:

If $g = x^m$ then the following cycles in K_1^R yield a basis for A_1

$$\begin{aligned} x^m \varepsilon_x \quad \text{and} \quad x^{m-j-1} d_j \varepsilon_x \quad \text{for } 1 \leq j \leq m-1, \\ y^{m-j-1} d_j \varepsilon_y \quad \text{for } 0 \leq j \leq m-1, \quad \text{and} \\ (-1)^{m-1} z^{m-1} d_{m-1} \varepsilon_z + x d_{m-2} \varepsilon_y. \end{aligned}$$

If $g = x^{m-i}d_i$ for some i in the range $1, \dots, m-1$, then the following cycles in K_1^R yield a basis for A_1

$$\begin{aligned} x^{m-j-1} d_j \varepsilon_x \quad \text{for } 0 \leq j \leq m-1, j \neq i \\ y^{m-j-1} d_j \varepsilon_y \quad \text{for } 0 \leq j \leq m-1, \quad \text{and} \\ (-1)^{m-1} z^{m-1} d_{m-1} \varepsilon_z + x d_{m-2} \varepsilon_y. \end{aligned}$$

If $g = d_m$ then the following cycles in K_1^R yield a basis for A_1

$$\begin{aligned} x^{m-j-1} d_j \varepsilon_x \quad \text{for } 0 \leq j \leq m-1, \\ y^{m-j-1} d_j \varepsilon_y \quad \text{for } 0 \leq j \leq m-1, \quad \text{and} \\ d_m \varepsilon_z. \end{aligned}$$

From Theorem (2.4) it is known that R is of class $\mathbf{G}(r)$ with $\mu(\mathfrak{a}) - 3 \leq r$. To prove that equality holds, which is the claim in (b) and (c), it suffices to show that the kernel of δ has rank at least $(\mu(\mathfrak{a}) + 1) - (\mu(\mathfrak{a}) - 3) = 4$; see (2.1). To this end we first notice that the cycles $g\varepsilon_{xy}$, $g\varepsilon_{xz}$, and $g\varepsilon_{yz}$ yield linearly independent elements of A_2 . Assume towards a contradiction that they are not, then there exists an element $h\varepsilon_{xyz}$ in K_3^Q and elements q_1, q_2 , and q_3 in Q and not all in \mathfrak{n} with

$$\partial(h\varepsilon_{xyz}) - (q_1 g \varepsilon_{xy} + q_2 g \varepsilon_{xz} + q_3 g \varepsilon_{yz}) \in \mathfrak{a} K_2^Q.$$

That is, one has $zh - q_1 g \in \mathfrak{a}$, $yh + q_2 g \in \mathfrak{a}$, and $xh - q_3 g \in \mathfrak{a}$, and hence $h \notin \mathfrak{n}^m$ as $g \notin \mathfrak{a} + \mathfrak{n}^{m+1}$. Furthermore, the class of h is a socle element in S as one has $\mathfrak{n}h \subseteq \mathfrak{a} + Qg = \mathfrak{g}_m$. Thus, $h \in \mathfrak{g}_m$ or $h = qx^{m-1}y^{m-1}$ for some $q \in Q \setminus \mathfrak{n}$. In either case one has $h \in \mathfrak{n}^m$, which is a contradiction. Thus $g\varepsilon_{xy}$, $g\varepsilon_{xz}$, and $g\varepsilon_{yz}$ yield linearly independent elements in A_2 that clearly belong to the kernel of δ .

Finally we produce a fourth element in the kernel. For $g = x^n$ the element

$$f = y^{m-1} \varepsilon_{yz}$$

is clearly a cycle in K_2^R , and it is not a boundary. Indeed, if one had $f = \partial(h\varepsilon_{xyz}) = hx\varepsilon_{yz} - hy\varepsilon_{xz} + hz\varepsilon_{xy}$ for some homogeneous element $h \in R$, then it would have degree $m-2$ and one would have $hy = 0 = hz$ in R , which is impossible as \mathfrak{a} has generators of degree at least m . The products $(y^{m-j-1}d_j\varepsilon_y) \cdot f$ and $((-1)^{m-1}z^{m-1}\varepsilon_z + x d_{m-2}\varepsilon_y) \cdot f$ in K^R vanish by graded commutativity. Moreover, one has

$$\begin{aligned} (x^m \varepsilon_x) \cdot f &= x(x^{m-1}y^{m-1})\varepsilon_{xyz} = 0 \quad \text{and} \\ (x^{m-j-1}d_j\varepsilon_x) \cdot f &= x^{m-j-1}y^{j-1}(y^{m-j}d_j)\varepsilon_{xyz} = 0. \end{aligned}$$

Thus the homology class of f annihilates A_1 .

For $g = x^{m-i}d_i$ and $1 \leq i \leq m-1$ the element

$$f = y^{m-i}d_{i-1}\varepsilon_{xy} + (-1)^{i-1}y^{m-i-1}d_i\varepsilon_{yz}$$

is a cycle in K_2^R ; indeed one has

$$\begin{aligned} \partial(f) &= xy^{m-i}d_{i-1}\varepsilon_y - y^{m-(i-1)}d_{i-1}\varepsilon_x + (-1)^{i-1}y^{m-i}d_i\varepsilon_z + (-1)^i y^{m-i-1}zd_i\varepsilon_y \\ &= y^{m-i-1}((-1)^i zd_i + xyd_{i-1})\varepsilon_y \\ &= y^{m-(i+1)}d_{i+1} \\ &= 0, \end{aligned}$$

where the third equality follows from Lemma (3.2). An argument similar to the one above shows that f is not a boundary. The products $(y^{m-j-1}d_j\varepsilon_y) \cdot f$ in K^R vanish by graded commutativity. Moreover, one has

$$(x^{m-j-1}d_j\varepsilon_x) \cdot f = (-1)^{i-1}x^{m-j-1}d_jy^{m-i-1}d_i\varepsilon_{xyz}.$$

If $i > j$ holds, then the element $x^{m-j-1}d_jy^{m-i-1}d_i$ is 0 in R because it is divisible by g , which is a socle element in R . If one has $i < j$, then the element $x^{m-j-1}d_jy^{m-i-1}d_i$ is zero in R because it is divisible in Q by the generator $y^{m-j}d_j$ of \mathfrak{a} . Finally, one has

$$\begin{aligned} ((-1)^{m-1}z^{m-1}d_{m-1}\varepsilon_z + xd_{m-2}\varepsilon_y) \cdot f &= (-1)^{m-1}y^{m-i}d_{i-1}z^{m-1}d_{m-1}\varepsilon_{xyz} \\ &= (-1)^{m-1}y^{m-i-1}d_{i-1}z^{m-1}(yd_{m-1})\varepsilon_{xyz} \\ &= 0 \end{aligned}$$

in K^R , so the homology class of f annihilates A_1 .

For $g = d_m$ the element

$$f = d_{m-1}\varepsilon_{xy}$$

is evidently a cycle in K_2^R , and as above it is not a boundary. The products $(x^{m-j-1}d_j\varepsilon_x) \cdot f$ and $(y^{m-j-1}d_j\varepsilon_y) \cdot f$ in K^R vanish by graded commutativity. Finally one has,

$$(d_m\varepsilon_z) \cdot f = d_{m-1}d_m\varepsilon_{xyz} = 0,$$

as $g = d_m$ is a socle element of R . \square

REFERENCES

1. Luchezar L. Avramov, *A cohomological study of local rings of embedding codepth 3*, J. Pure Appl. Algebra **216** (2012), no. 11, 2489–2506. MR2927181
2. Luchezar L. Avramov and Evgeniy S. Golod, *On the homology algebra of the Koszul complex of a local Gorenstein ring*, Mat. Zametki **9** (1971), 53–58. MR0279157
3. Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), no. 1, 162–204. MR0961334
4. David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), no. 3, 447–485. MR0453723
5. Lars Winther Christensen and Oana Veliche, *Local rings of embedding codepth 3. Examples*, Algebr. Represent. Theory **17** (2014), no. 1, 121–135. MR3160716
6. Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, *Linkage classes of grade three perfect ideals*, in preparation.
7. Jessica Ann Faucett, *Expanding the socle of a codimension 3 complete intersection*, Rocky Mountain J. Math., online September 2016.

8. Junzo Watanabe, *A note on Gorenstein rings of embedding codimension three*, Nagoya Math. J. **50** (1973), 227–232. MR0319985
9. Jerzy Weyman, *On the structure of free resolutions of length 3*, J. Algebra **126** (1989), no. 1, 1–33. MR1023284

TEXAS TECH UNIVERSITY, LUBBOCK, TX 79409, U.S.A.

E-mail address: `lars.w.christensen@ttu.edu`

URL: `http://www.math.ttu.edu/~lchrste`

NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, U.S.A.

E-mail address: `o.veliche@neu.edu`

UNIVERSITY OF CONNECTICUT, STORRS, CT 06269, U.S.A.

E-mail address: `jerzy.weyman@uconn.edu`

URL: `http://www.math.uconn.edu/~weyman`